A GENERALIZED MOMENT PROBLEM

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ABSTRACT

Let $\{\lambda_n\}$ $(n \geqq 0)$ satisfy (1.1) we are considering the following problems: What are the necessary and sufficient conditions on a sequence $\{\mu_n\}(n \geqq 0)$ in order that it should possess the representation (1.2) where $\alpha(t)$ is of bounded variation or the representation (1.3) where $f(t) \in L_M[0,1]$ or $f(t)$ is essentially bounded.

1. **Introduction and definitions.** Let the sequence $\{\lambda_i\}$ $(i \geqq 0)$ possess the following properties:

$$(1.1) \qquad 0 \leqq \lambda_0 < \lambda_1 < \cdots < \lambda_n \uparrow \infty, \quad \sum_{i=1}^{\infty} 1/\lambda_i = \infty .$$

We shall discuss the following problems: What are the conditions, necessary and sufficient, on a sequence $\{\mu_n\}$ $(n \geqq 0)$ in order that it should possess the representation

$$(1.2) \qquad \mu_n = \int_0^1 t^{\lambda_n} d\alpha(t) \qquad n = 0, 1, 2, \cdots$$

where $\alpha(t)$ is of bounded variation in $[0,1]$.

What are the conditions, necessary and sufficient, on a sequence $\{\mu_n\}$ $(n \geqq 0)$ in order that it should possess the representation:

$$(1.3) \qquad \mu_n = \int_0^1 t^{\lambda_n} f(t)\, dt \qquad n = 0, 1, 2, \cdots$$

where $f(t)$ belongs to a given class of functions integrable over $[0,1]$.

Hausdorff [3] gave the answer to the first problem in the case $\lambda_0 = 0$. Endl [2] solved the same problem in the case $\lambda_0 > 0$ and the function $\alpha(t)$ is nondecreasing in $[0,1]$.

Schoenberg [9] obtains the same solution as Hausdorff [3] in another way and we shall use in this paper some of his results.

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Let A be an infinite matrix of real numbers

$$A = \| a_{nm} \| \qquad n = 0,1,2,\cdots \quad m = 1,2,\cdots$$

where $a_{i1} = 1$ $i = 0,1,2,\cdots$.

Denote

$(i_1,\cdots,i_m) = \det \| a_{i_k,r} \|$ $0 \leqq i_1 < \cdots < i_m$, $r = 1,\cdots,m$ (if $m = 1$ $(i_1) = a_{i_1,1}$).

Let us assume that $(i_1,\cdots,i_m) > 0$ for every $0 \leqq i_1 < i_2 < \cdots < i_m$.

For a sequence $\{\mu_n\}$ $(n \geqq 0)$ define:

$$D^k\mu_s = \begin{vmatrix} \mu_s, a_{s,1}, \cdots, a_{s,k} \\ \vdots \\ \mu_{s+k}, a_{s+k,1}, \cdots, a_{s+k,k} \end{vmatrix}$$

(when $k = 0$ $D^0\mu_s = \mu_s$).

We denote after Schoenberg [9]

$$\lambda_{nm} = \frac{(0, m+1,\cdots,n)}{(m+1,\cdots,n)(m,\cdots,n)} D^{n-m}\mu_m \qquad 0 \leqq m < n = 1,2,\cdots \tag{1.4}$$

and

$$\lambda_{nn} = \frac{(0)}{(n)}\mu_n = \mu_n, \qquad t_{nm} = \frac{(1, m+1,\cdots,n)}{(0, m+1,\cdots,n)} \quad 0 \leqq m < n = 1,2,\cdots$$

and $t_{nn} = 1$.

We shall use the function $\{\phi_n(x)\}$ $(n \geqq 0)$ defined by Schoenberg [9] where it was proved that the functions $\phi_n(x)$ are continuous convex functions and that $0 = t_{n0} < t_{n1} < \cdots < t_{nn} = 1$.

If A is an infinite Vandermonde, i.e.

$$A = \| a_{nm} \|, \; a_{nm} = (\lambda_n)^{m-1} \qquad n = 0,1,2,\cdots \quad m = 1,2,\cdots$$

where $\{\lambda_i\}$ satisfies Condition (1.1) then it was shown in Schoenberg [9] that

$$\phi_n(x) = x^{(\lambda_n - \lambda_0)/(\lambda_1 - \lambda_0)} \qquad \text{for } n \geqq 0 \tag{2.1}$$

and that

$$\lambda_{nm} = (-1)^{n-m}(\lambda_{m+1} - \lambda_0)\cdot \cdots \cdot(\lambda_n - \lambda_0)[\mu_m,\cdots,\mu_n],$$

where

$$[\mu_m,\cdots,\mu_n] \equiv \sum_{i=m}^{n} \frac{\mu_i}{(\lambda_i - \lambda_m)\cdot \cdots \cdot(\lambda_i - \lambda_{i-1})(\lambda_i - \lambda_{i+1})\cdot \cdots \cdot(\lambda_i - \lambda_n)} \tag{2.2}$$

(see also Jakimovski [5] (11.3)).

2. **The main results.** First we shall generalize Hausdorff's solutions [3] by solving the first problem for $\lambda_0 > 0$.

THEOREM 2.1. *Let* $\{\lambda_i\}$ $(i \geqq 0)$ *satisfy Condition* (1.1). *The sequence* $\{\mu_n\}$ $(n \geqq 0)$ *possesses the representation* (1.2), *if, and only if*:

$$\sup_{n\geqq 0} \sum_{m=0}^{n} \lambda_{m+1} \cdot \cdots \cdot \lambda_n \left| [\mu_m, \cdots, \mu_n] \right| \equiv H < \infty . \tag{2.3}$$

Let $M(u)$ be an even, convex continuous function satisfying 1. $M(u)/u \to 0 (u \to 0)$, 2. $M(u)/u \to \infty$ $(u \to \infty)$. Denote by $L_M[0,1]$ the class of functions integrable over $[0,1]$ such that $\int_0^1 M[f(x)]\,dx < \infty$. $L_M[0,1]$ is the Orlicz class related to $M(u)$. (See [6]).

If we take $M(u) = |u|^p$ $p > 1$, $L_M[0,1]$ is the space $L^p[0,1]$.

The Orlicz class $L_M[0,1]$ is not necessarily a linear space (see [6] Theorem 8.2).

Denote by $M[0,1]$ the space of all functions essentially bounded in $[0,1]$.

THEOREM 2.2. *Suppose that* $\{\phi_n(x)\}$ $(n \geqq 0)$ *spans the space* $C[0,1]$ *in the supremum norm. The sequence* $\{\mu_n\}$ $(n \geqq 0)$ *possesses the representation*:

$$\mu_n = \int_0^1 \phi_n(t) f(t)\,dt \qquad n = n = 0,1,2,\cdots \tag{2.4}$$

where: (i) $f(t) \in L_M[0,1]$ if, and only if,

$$\sup_{n\geqq 0} \sum_{m=0}^{n} \left[\int_0^1 \lambda_{nm}(t)\,dt\right] M \left[\frac{\lambda_{nm}}{\int_0^1 \lambda_{nm}(t)\,dt}\right] \equiv H < \infty . \tag{2.5}$$

(ii) $f(t) \in M[0,1]$ if, and only if,

$$\sup_{\substack{0\leqq m\leqq n\\ n\geqq 0}} \frac{|\lambda_{nm}|}{\int_0^1 \lambda_{nm}(t)\,dt} \equiv H < \infty . \tag{2.6}$$

$$(\lambda_{nm}(t) = \frac{(0, m+1, \cdots, n)}{(m+1, \cdots, n)\,(m, \cdots, n)} D^{n-m} \phi_m(t) \text{ for } 0 \leqq m < n = 1,2,\cdots$$

and $\lambda_{nn}(t) = \phi_n(t)$, by [9] Theorem 8.1 $\lambda_{nm}(t) \geqq 0$ for $0 \leqq t \leqq 1$, $0 \leqq m \leqq n = 0,1,2,\cdots)$.

THEOREM 2.3. *Let* $\{\lambda_i\}$ $(i \geqq 0)$ *satisfy* (1.1) *with* $\lambda_0 = 0$. *The sequence* $\{\mu_n\}$ $(n \geqq 0)$ *possesses the representation* (1.3) *where*: (i) $f(t) \in L_n[0,1]$ if, *and only if,*

$$\sup_{n\geqq 0} \sum_{m=0}^{n} \left[\int_0^1 (-1)^{n-m} \lambda_{m+1} \cdot \cdots \cdot \lambda_n [t^{\lambda_m}, \cdots, t^{\lambda_n}]\,dt\right] M \left[\frac{[\mu_m, \cdots, \mu_n]}{\int_0^1 [t^{\lambda_m}, \cdots, t^{\lambda_n}]\,dt}\right] \equiv H < \infty \tag{2.7}$$

(ii) $f(t) \in M[0,1]$ *if, and only if,*

$$\sup_{\substack{0 \leq m \leq n \\ n \geq 0}} \left| \frac{[\mu_m, \cdots, \mu_n]}{\int_0^1 [t^{\lambda_m}, \cdots, t^{\lambda_n}]\, dt} \right| \equiv H < \infty . \tag{2.8}$$

By (2.1), and Müntz theorem (see [7] Theorem 2.8.1), Theorem 2.3 in the case $\lambda_1 = 1$ follows from Theorem 2.2.

For $\lambda_i = i$, $i = 0, 1, 2, \cdots$ and $M(u) = |u|^p$ $1 < p < \infty$, Theorem 2.3 (i) is Hausdorff's Theorem III [4] and for $\lambda_i = i$, $i = 0, 1, 2, \cdots$, Theorem 2.3 (ii) is Hausdorff's Theorem IV [4]. For $\lambda_i = i, i = 0, 1, \cdots$ Theorem 2.3 (*i*) was proved by Berman [1].

3. Proofs of the Theorems.

Proof of Theorem 2.1. We have to prove the theorem only in the case $\lambda_0 > 0$ since for $\lambda_0 = 0$ this is Hausdorff's Theorem VI [3].

First we prove the necessity.

Define the sequence $\{\tilde{\mu}_n\}$, $\{\tilde{\lambda}_n\}$ $(n \geqq 0)$ by the equations

$$\tilde{\lambda}_0 = 0,\ \tilde{\mu}_0 = \alpha(1) - \alpha(0),\ \tilde{\lambda}_n = \lambda_{n-1},\ \tilde{\mu}_n = \mu_{n-1} \quad (n \geqq 1) \tag{3.1}$$

by (1.2) and (3.1) we have

$$\tilde{\mu}_n = \int_0^1 t^{\tilde{\lambda}_n} d\alpha(t) \qquad n = 0, 1, 2, \cdots \tag{3.2}$$

Hence by Hausdorff's Theorem VI [3]

$$\sup_{n \geqq 0} \sum_{m=0}^{n} \tilde{\lambda}_{m+1} \cdot \cdots \cdot \tilde{\lambda}_n \left| [\tilde{\mu}_m, \cdots, \tilde{\mu}_n] \right| \equiv L < \infty . \tag{3.3}$$

By an easy calculation we get from (2.3) that for $1 \leqq m \leqq n = 1, 2, \cdots$

$$[\tilde{\mu}_m, \cdots, \tilde{\mu}_n] = [\mu_{m-1}, \cdots, \mu_{n-1}] . \tag{3.4}$$

Therefore by (3.3) we get

$$\sup_{n \geqq 0} \sum_{m=0}^{n} \lambda_{m+1} \cdot \cdots \cdot \lambda_n \left| [\mu_m, \cdots, \mu_n] \right| \equiv H \leqq L < \infty .$$

Thus we prove (2.2).

In order to prove the sufficiency let us define the sequences $\{\tilde{\mu}_n\}$, $\{\tilde{\lambda}_n\}$ $(n \geqq 0)$ by (3.1), with one exception, $\tilde{\mu}_0$ is arbitrary.

By Hausdorff (7) [3] we get:

$$\sum_{m=0}^{n} (-1)^{n-m} \tilde{\lambda}_{m+1} \cdot \cdots \cdot \tilde{\lambda}_n [\tilde{\mu}_m, \cdots, \tilde{\mu}_n] = \tilde{\mu}_0 .$$

(by (3.4))

$$(-1)^n\tilde{\lambda}_1\cdot\dots\cdot\tilde{\lambda}_n[\tilde{\mu}_0,\cdots,\tilde{\mu}_n] = \tilde{\mu}_0 - \sum_{m=0}^{n-1}(-1)^{n-1-m}\lambda_{m+1}\cdot\ \cdots\cdot\lambda_{n-1}[\mu_m,\cdots,\mu_{n-1}].$$

Hence by (2.2)

$$\tilde{\lambda}_1\cdot\dots\cdot\tilde{\lambda}_n\,|\,[\tilde{\mu}_0,\cdots,\tilde{\mu}_n]\,| \leqq |\,\tilde{\mu}_0\,| + H. \tag{3.5}$$

By (2.2), (3.4) and (3.5) we get for every $n \geqq 0$:

$$\sum_{m=0}^{n}\tilde{\lambda}_{m+1}\cdot\dots\cdot\tilde{\lambda}_n\,|\,[\tilde{\mu}_m,\cdots,\tilde{\mu}_n]\,| \leqq K < \infty$$

where K does not depend on n.

Hence by Hausdorff's Theorem VI [3]:

$$\tilde{\mu}_n = \int_0^1 t^{\tilde{\lambda}_n}d\alpha(t) \qquad n = 0,1,2,\cdots \tag{3.6}$$

where $\alpha(t)$ is of bounded variation in $[0,1]$.

Now by (3.1) and (3.6)

$$\mu_n = \int_0^1 t^{\lambda_n}d\alpha(t) \qquad n = 0,1,2,\cdots \qquad \text{Q.E.D.}$$

Proof of Theorem 2.2. (i) By corollary 8.1 of Schoenberg the proof is as that of Berman [1], but now the results of Schoenberg [9] are used. (ii) In order to prove necessity, let us assume that $\{\mu_n\}$ $(n \geqq 0)$ possesses the representation (2.4) where $f(t) \in M[0,1]$. We have

$$|\,\lambda_{nm}| \leqq \int_0^1 \lambda_{nm}(t)\,|f(t)|\,dt \leqq H\int_0^1 \lambda_{nm}(t)\,dt$$

where $H = \operatorname{ess\,sup}_{0\leqq t\leqq 1}|f(t)|$.

Thus we proof necessity.

We prove now sufficiency. By (2.6) and since

$$\sum_{m=0}^{n}\lambda_{nm}(t) = \phi_0(t) = 1 \qquad \text{(see [9] p. 607 (8.23))}, \tag{3.7}$$

we get

$$\sum_{m=0}^{n}|\,\lambda_{nm}| \leqq H\int_0^1 \sum_{m=0}^{n}\lambda_{nm}(t)\ dt = H.$$

Hence by Corollary 8.1 of Schoenberg [9], $\{\mu_n\}$ $(n \geqq 0)$ possesses the representation

(3.8) $$\mu_n = \int_0^1 \phi_n(t)\,d\alpha(t) \qquad n = 0,1,2,\cdots$$

where $\alpha(t)$ is of bounded variation in $[0,1]$, and if we define $\alpha_n(x)$ by:

$$\alpha_n(0) = 0,\ \alpha_n(x) = \sum_{t_{nm}\leqq x} \lambda_{nm} \qquad 0 < x \leqq 1,$$

then there exists a subsequence $\{n_i\}$ $(i \geqq 0)$ such that $\lim \alpha_{n_i}(x) = \alpha(x)$ for $0 \leqq x \leqq 1$.

Let x, y, $0 \leqq x < y \leqq 1$, there exist r, s satisfying

$$t_{n,r} \leqq x < t_{n,r+1}, \quad t_{n,s} \leqq y < t_{n,s+1}$$

(r, s depend on n).

Now $\left|\alpha_n(y) - \alpha_n(x)\right| \leqq \sum_{m=r+1}^{s} \left|\lambda_{nm}\right| \leqq H\left[\sum_{m=r+1}^{s} \int_0^1 \lambda_{nm}(t)\,dt\right]$

hence for every $n \geqq 0$: $\dfrac{\left|\alpha_n(y) - \alpha_n(x)\right|}{\sum_{m=r+1}^{s} \int_0^1 \lambda_{nm}(t)\,dt} \leqq H.$

We have $\lim_{i\to\infty}\{(\alpha_{n_i}(y) - \alpha_{n_i}(x)\} = \alpha(y) - \alpha(x)$. Since $\{\phi_n(x)\}$ $(n \geqq 0)$ spans $C[0,1]$ we have by [9] Theorem 8.1 and Corollary 8.1 that the solution of the moment problem is unique. By Helly's theorem every sequence $\{n_i\}$ $(i \geqq 0)$ has a subsequence $\{k_j\}$ $(j \geqq 0)$ such that $\lim_{j\to\infty} \sum_{tk_j,m\leqq x} \int_0^1 \lambda_{k_j,m}(t)\,d\alpha(t) = \alpha(x)$ for each point $t = x$ where $\alpha(t)$ is continuous. Hence $\lim_{n\to\infty} \sum_{t_{nm}\leqq x}\int_0^1 \lambda_{nm}(t)d\alpha(t) = \alpha(x)$ for each point $t = x$ where $\alpha(t)$ is continuous and we obtain

$$\lim_{i\to\infty} \sum_{m=r+1}^{s} \int_0^1 \lambda_{n_i,m}(t)\,dt = y - x.$$

Therefore $\dfrac{\left|\alpha(y) - \alpha(x)\right|}{y-x} \leqq H$ for any two points x, y, $0 \leqq x < y \leqq 1$, hence $\alpha(x) = c + \int_0^x f(t)\,dt$ where $f(t) \in M[0,1]$ and by (3.8):

$$\mu_n = \int_0^1 \phi_n(t) f(t)\,dt \qquad n = 0, 1, 2, \cdots. \qquad \text{Q.E.D}$$

Proof of Theorem 2.3. The proof of the necessity is similar to that of Theorem 2.2 using, instead of (3.7) formula (11) p. 46 of Lorentz [7]

$$\sum_{m=0}^{n} (-1)^{n-m} \lambda_{m+1} \cdot \cdots \cdot \lambda_n [t^{\lambda_m}, \cdots, t^{\lambda_n}] = 1 \quad \text{for } 0 \leqq t \leqq 1.$$

We prove now the sufficiency. As in the proof of Theorem 2.2 we get

$$\sup_{n \geq 0} \sum_{m=0}^{n} \lambda_{m+1} \cdot \cdots \cdot \lambda_n \left| [\mu_m, \cdots, \mu_n] \right| \equiv K < \infty$$

Define functions $\alpha_n(x)$ by:

$$\alpha_n(0) = 0 \qquad \alpha_n(x) = \sum_{t_{nm}^{1/\lambda_1} \leq x} (-1)^{n-m} \lambda_{m+1} \cdot \cdots \cdot \lambda_n [\mu_m, \cdots, \mu_n] \qquad 0 < x \leq 1$$

and we get by Schoenberg [9] that for every $k \geq 0$

$$\int_0^1 t^{\lambda_k} d\alpha_n(t) = \sum_{m=0}^{n} t_{nm}^{\lambda_k/\lambda_1} (-1)^{n-m} \lambda_{m+1} \cdot \cdots \cdot \lambda_n [\mu_m, \cdots, \mu_n] \to \mu_k$$

as $n \to \infty$. Using Helly's theorem (see [10] p. 29), since $\alpha_n(x)$ are of variations uniformly bounded in $[0,1]$ we get $\lim \alpha_{n_i}(x) = \alpha(x)$ for $0 \leq x \leq 1$. By Helly-Bray theorem (see [10] p. 31)

$$\mu_k = \int_0^1 t^{\lambda_k} d\alpha(t) \qquad k = 0, 1, 2, \cdots.$$

We conclude the proof as in Theorem 2.2. Q.E.D.

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